ANALYSIS III BACKPAPER EXAMINATION

Total marks: 100

(1) State the Inverse Function Theorem and the Implicit Function Theorem. Let $F = (f_1, f_2) : \mathbb{R}^3 \to \mathbb{R}^2$ be a continuously differentiable function. Let $a \in \mathbb{R}^3$ such the total derivative $dF(a)$ at a, has rank 2. Show that there exists a continuously differentiable function $f_3 : \mathbb{R}^3 \to \mathbb{R}$, such that the function $\Phi = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ has a continuously differentiable inverse when restricted to an open neighborhood of a. (5+5+10 = 20 marks)

(2) Prove that if A is any bounded subset of $\mathbb{R}^m$, and if B is any subset of $\mathbb{R}^n$ of volume zero, then $A \times B$ is a subset of $\mathbb{R}^{m+n}$ of volume zero. Using this, prove that the product of any two Jordan regions $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ is Jordan. State Fubini's Theorem (any version). (6+6+8 = 20 marks)

(3) State the Change of Variables Formula (the most general version that you know). Consider the n-dimensional solid sphere of radius r, $B^n(r) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \ldots + x_n^2 \leq r^2\}$. Let $V_n(r) := \int \cdots \int_{B_n(r)} dx_1 \ldots dx_n$ denote the volume of $B_n(r)$. Using the above theorem, prove that $V_n(r) = r^n V_n(1)$, and then prove that $V_n(1) = \frac{2\pi}{n} V_{n-2}(1)$ if $n \geq 3$. (8+5+7 = 20 marks)

(4) Let C be the triangle in the plane with vertices $(0,0), (1,0)$ and $(1,1)$. Compute the line integral $\int_C x^3 dx - x^2 y dy$ (going over C once in the anticlockwise direction). Write down the statement of Green's theorem, and using it, compute the line integral again by computing the relevant double integral. (6+8+6 = 20 marks)

(5) State the classical form of the Divergence theorem of Gauss. Let E be the solid unit cube in $\mathbb{R}^3$ with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)$. Consider a 2-form in $\mathbb{R}^3$ with component vector field $F = (2xy, 3ye^z, xsin(z))$. Compute both sides of the theorem of Gauss for this F (you have to consider the boundary of the cube with outward pointing normal vector). (8+6+6 = 20 marks)